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MATERIALS WITH PRESCRIBED CONSTITUTIVE PARAMETERS: AN INVERSE HOMOGENIZATION PROBLEM

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Abstract—This paper deals with the construction of materials with arbitrary prescribed positive semi-definite constitutive tensors. The construction problem can be called an inverse problem of finding a material with given homogenized coefficients. The inverse problem is formulated as a topology optimization problem, i.e. finding the interior topology of a base cell such that cost is minimized and the constraints are defined by the prescribed constitutive parameters. Numerical values of the constitutive parameters of a given material are found using a numerical homogenization method expressed in terms of element mutual energies. Numerical results show that arbitrary materials, including materials with Poisson's ratio -1.0 and other extreme materials, can be obtained by modelling the base cell as a truss structure. Furthermore, a wide spectrum of materials can be constructed from base cells modelled as continuous discs of varying thickness. Only the two-dimensional case is considered in this paper but formulation and numerical procedures can easily be extended to the three-dimensional case.

1. INTRODUCTION

The aim of this paper is to formulate and implement a procedure to construct materials with any prescribed constitutive tensor only restricted to be positive semi-definite. The development is restricted to linear elastic materials and small deformation kinematics in both the macroscopic and the microscopic scale. Only materials with a periodic microstructure will be considered. This implies that the macroscopic behaviour or effective average elastic constants can be found by analysing the microstructure represented by base cells.

Inspiration to do this work comes partly from a recent paper by Bendsøe *et al.* (1993), in which it is shown that a global lower bound on compliance optimization is obtained by using a locally orthotropic material with the directions of orthotropy following the directions of principal strains. The rigidity tensor of these "optimal" materials has two zero eigenvalues, which means that the materials are only stable subject to the loading condition for which they are designed and will collapse for other loading conditions. This paper will show that it is possible to construct such extreme materials.

Another source of inspiration to do this work is the search for materials with Poisson's ratio -1.0. Several papers suggest different kinds of structures and materials which exhibit negative Poisson's ratios. Examples are: Almgren (1985), where a mechanism consisting of struts, sliding collars and springs is proposed; Lakes (1987) has patented a foam structure with negative Poisson's ratio; Milton (1992) who obtains negative Poisson's ratios considering a microstructure composed of a soft matrix phase and an almost rigid hexagonal inclusion phase connected by small hinges; Phan-Tien and Karihaloo (1993) who suggest a forked microstructure in a weak matrix material. The common approach of these papers is to start with a microstructure which intuitively could exhibit negative Poisson's ratio and then modifying connections and topology to make the Poisson's ratio as low as possible.

In this paper an inverse approach is used to construct materials with arbitrary properties. It is expected that many differently composed materials will exhibit the same constitutive behaviour, so, for practical reasons, the goal will be to construct the lightest and possibly simplest material with the given constitutive parameters. The construction problem

is therefore formulated as an optimization problem, where the cost function to be minimized is the weight, the constraints are the given constitutive parameters, and the design variables define the composition and topology of the material. A similar inverse approach is proposed in Autio *et al.* (1993), but there the possible design variables are limited to ply orientation and/or ply thicknesses of laminates, which of course puts limits on the attainable constitutive properties. On the other hand, this procedure produces materials that can be manufactured easily. A similar problem of constructing materials with given constitutive parameters was handled in a recent paper by Milton and Cherkaev (1993) where it is shown that it is possible to construct all thermodynamically admissible materials by layering of infinitely strong and infinitely weak materials.

As mentioned earlier, the "optimal" materials proposed in Bendsøe *et al.* (1993) are unstable and will perform mechanism-type motion subject to loads differing from the design load. It is not obvious that such extreme materials can be constructed, but assuming that they can, a promising approach will be to use a truss-like material which can exhibit mechanism behaviour. As a starting guess for the optimization problem, we use a ground structure as used in topology optimization of trusses, with individual bar areas as design variables. Another approach that we use is to model the material as a continuous disc of varying thickness. Here, the design variables are the thicknesses of the individual finite elements used to discretize the disc, and again (near) mechanisms can occur if the lower limit on the design variables is chosen small compared with the average thickness.

The periodic materials we consider are described by the base cell which is the smallest repetitive unit of a material and calculation of the effective constitutive parameters (homogenization) can be performed by analysing the base cell only. This procedure is described several times in literature, some examples are: Bensoussan *et al.* (1978); Sanchez-Palencia (1980); Bakhalov and Panasenko (1989); Aboudi (1991); Cioranescu and Saint Jean Paulin (1986). Analytical determination of homogenized parameters is possible in some cases of simple periodic microstructures. Considering more complicated microstructures, analytical determination of material behaviour becomes an impossible task and numerically based methods [finite-element based methods proposed in e.g. Bourgat (1977) and Guedes and Kikuchi (1990), or numerical boundary integral equation methods proposed in e.g. Greenbaum *et al.* (1993)] must therefore be used.

In this paper we use the finite-element based homogenization formulae (Bourgat, 1977; Guedes and Kikuchi, 1990) in terms of element mutual energies, which makes the inverse problem more suited for optimization. The optimization problem is formulated as a multiple load, minimum weight problem and is solved by a modified version of the optimality criterion method proposed in Zhou and Rozvany (1993).

This paper is divided into five sections. In Section 2 we describe the notation used for material properties and discuss behavior of extreme materials. In Section 3 a brief overview of homogenization theory is given expressed in terms of element mutual energies. The optimization procedure is described in Section 4 and the numerical results obtained from microstructures modelled by trusses and continuous discs are presented in Section 5 where also the manufacturability of the proposed materials is discussed.

2. MATERIAL PROPERTIES

In this paper we are concerned with the general constitutive laws in two-dimensional linear elasticity. The constitutive law is defined by the positive semi-definite constitutive tensor E_{ijkl} and is given by

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \tag{1}$$

where σ_{ij} and ε_{kl} are the stress and strain tensors in the reference coordinate system. For convenience, reference to the constitutive tensor will, in the following, be given in standard engineering notation such that E_{ijkl} is written in matrix form as

$$[E] = \begin{bmatrix} E_{1111} & E_{1122} & E_{1112} \\ E_{1122} & E_{2222} & E_{2212} \\ E_{1112} & E_{2212} & E_{1212} \end{bmatrix}. \tag{2}$$

Due to symmetry of the constitutive tensor there are only six independent constitutive parameters in the two-dimensional case but for isotropic, orthotropic or otherwise symmetric materials, some of the six constitutive parameters E_{ijkl} will be dependent or zero.

The materials handled in this paper can be modelled as discs (only inplane forces and displacements) subject to plane stress condition, i.e. the constitutive matrix for an isotropic material can be written as

$$[E]_{\text{plane stress}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{3}$$

where E is the modulus of elasticity and ν is Poisson's ratio defined in the interval $]-1, 1[$. In Lakes (1987) it is remarked that materials with Poisson's ratios approaching -1.0 will be very tough in practice because of the singularity in the factor $E/(1-\nu^2)$ in eqn (3). In this paper, materials are treated only qualitatively, i.e. the factor $E/(1-\nu^2)$ is considered a constant equal to unity. By simple linear scaling, all desired stiffnesses can be obtained.

A common approach in structural optimization with anisotropic materials has been to optimize a structure such that a small number of given materials are utilized in the best possible way by varying density, lay-up and orientation throughout the structure. In Bendsøe *et al.* (1993), a global lower bound for single load compliance optimization is obtained by a free parametrization of the constitutive tensor. A measure of cost using invariants of the constitutive tensor ensures that the optimal solution is independent of the choice of reference frame. The only constraint imposed is positive semi-definiteness of the constitutive tensor, for physical reasons. The resulting "optimal" material in the frame of principal stresses is given by the constitutive matrix

$$[E]_{\text{opt}} = \frac{\varrho}{\varepsilon_I^2 + \varepsilon_{II}^2} \begin{bmatrix} \varepsilon_I^2 & \varepsilon_I\varepsilon_{II} & 0 \\ \varepsilon_I\varepsilon_{II} & \varepsilon_{II}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

where ε_I and ε_{II} are the principal strains for the loading case considered and ϱ is a scaling factor (the resource density). The "optimal" material is an orthotropic material and is oriented in the direction of the principal strains. The constitutive tensor has two zero eigenvalues, which means that the material is only marginally stable and will collapse in mechanism-type motion for all load cases that are not linear scalings of the one for which it is designed.

Numerical results in Section 5 will show that truss-like materials can model "optimal" materials for all combinations of ε_I and ε_{II} in eqn (4). Furthermore, numerical experiments show that for many combinations of ε_I and ε_{II}, the "optimal" material parameters [eqn (4)] can be obtained by modelling the base cells as continuous discs of varying thickness.

3. HOMOGENIZATION

Computation of effective constitutive parameters of complex materials can be done by using a numerical implementation of the standard homogenization method [see e.g. Bourgat (1977) or Guedes and Kikuchi (1990)]. The homogenization method can be used on spatial repetitive materials, where periodicity is represented by a base cell which is very small

compared with the size of the structural body. Figures 1 and 2 show two examples of repetitive bodies; Fig. 1 shows a typical composite structure consisting of a strong (fibre) and a weak (matrix) material, and Fig. 2 shows a repetitive truss structure. Common to both structures is the high level of heterogeneity, which makes discretization of the full structure impossible and calls for the use of "smoothing" methods like homogenization.

Before describing the homogenization procedure, it is necessary to consider the modelling of trusses as continuum structures. Primarily, it is assumed that the truss structure is a continuum with holes and that none of the holes intersects the cell boundaries. These assumptions make it possible to apply the homogenization formulae. The homogenization formulae are solved by a finite element approach where the individual bar members in the truss-like cell are treated as two-node continuum elements that only have stiffness in the longitudinal direction and that have zero shear stiffness. Treating the bar elements this way, the usual stress–strain relations from two-dimensional elasticity (Cook *et al.*, 1989) can be applied. This has the important implication that the same software can be used in finding the homogenized coefficients for truss-like and continuum-like materials, the only differences being the number of elements, the strain-displacement matrices and the local constitutive laws for the truss- and continuum-type elements. For certain geometries a direct asymptotic study of the base cell can lead directly to truss models [e.g. Bakhalov and Panasenko (1989) and Cioranescu and Saint Jean Paulin (1986)], showing that truss models are asymptotically exact.

The material for which we want to find the constitutive parameters is assumed to consist of a periodically repeated microstructure. The microstructure is described by base cells as shown in Figs 1 and 2, which are rectangular in $\mathbb{R}^2$ and defined as

$$Y =]0, y_1^0[\times]0, y_2^0[\tag{5}$$

where y_1^0 and y_2^0 are the horizontal and vertical length of the base cell, respectively.

Following standard homogenization procedure, the displacement field u can be expanded in an asymptotic series

$$u = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \cdots, \quad y = x/\varepsilon \tag{6}$$

where the involved functions are dependent on the global macroscopic variable x and the local microscopic variable y. If only first order terms of the asymptotic expansion are considered, the global properties of the material are given by

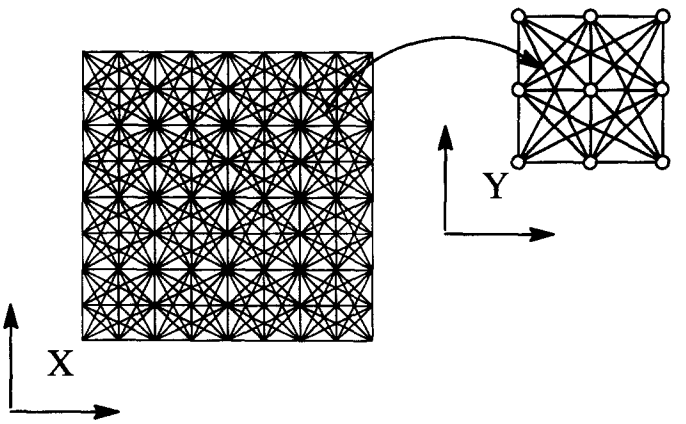
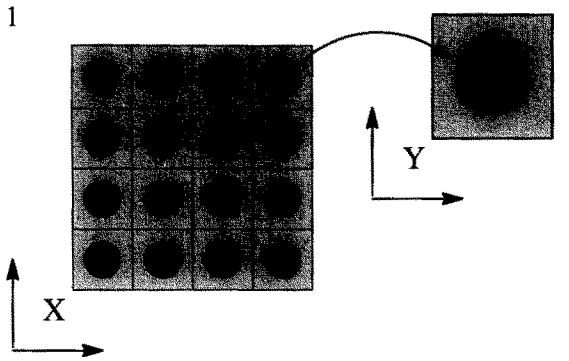
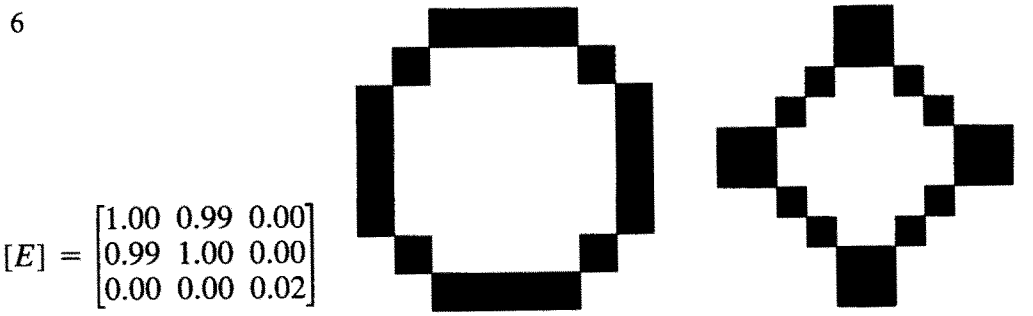
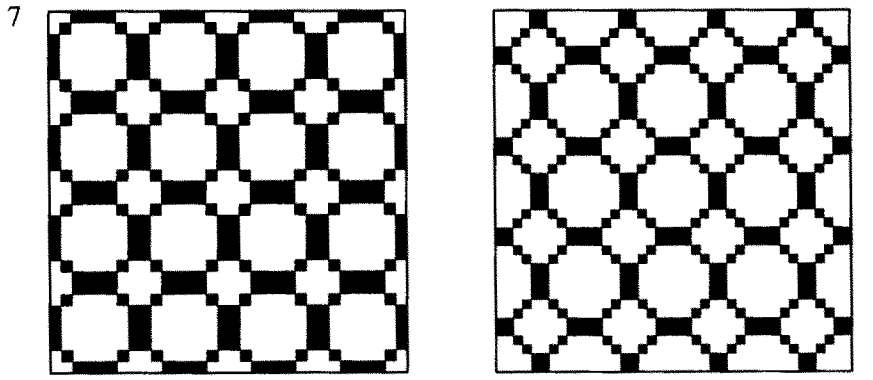


Fig. 2. Composite structure composed of truss-like material (small circles illust[illegible]







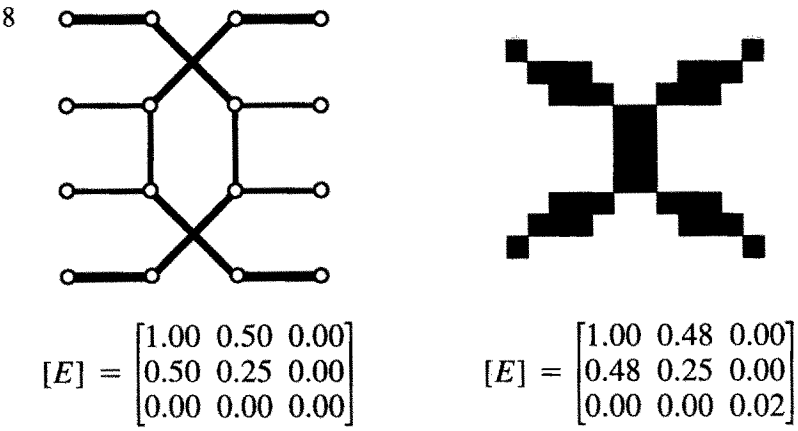


Fig. 1. Composite structure composed of continuum-like material.

Fig. 6. "Optimal" materials with $\varepsilon_I = 1.0$ and $\varepsilon_{II} = 1.0$ or Poisson's ratio 1.0 materials. Weights are 3.82 and 3.86 and maximum thicknesses are 15.0 and 20.0, respectively.

Fig. 7. Periodic materials with $\varepsilon_I = 1.0$ and $\varepsilon_{II} = 1.0$ or Poisson's ratio 1.0 materials composed from base cells in Fig. 6 (note similarity with truss-like materials in Fig. 5).

Fig. 8. "Optimal" materials for $\varepsilon_I = 1.0$ and $\varepsilon_{II} = 0.5$. Weights are 2.25 and 4.77, respectively and maximum thickness of the continuum-like material is 30.0.

$$E^H_{ijkl} = \frac{1}{Y}\int_Y E_{ijpq}(\varepsilon^{0(kl)}_{pq} - \varepsilon^{*(kl)}_{pq})\,\mathrm{d}Y \tag{7}$$

where $\varepsilon^{*(kl)}_{pq}$ is the periodic solution to the variational type problem

$$\int_Y E_{ijpq}\varepsilon^{*(kl)}_{pq}\frac{\partial v_i}{\partial y_j}\,\mathrm{d}Y = \int_Y E_{ijpq}\varepsilon^{0(kl)}_{pq}\frac{\partial v_i}{\partial y_j}\,\mathrm{d}Y \quad \forall v \in V$$

$$V = \{v : v \text{ is } Y\text{-periodic}\} \tag{8}$$

and $\varepsilon^{0(kl)}_{pq}$ is the three cases of unit prestrain as sketched in Fig. 3.

Equation (8) is the weak form of the standard elasticity equation applied to a base cell with periodic boundary conditions. The equation is solved by finite-element analysis of the base cell subject to three independent cases of prestrain given by $\varepsilon^{0(kl)}_{pq}$. Finally, substituting the solutions $\varepsilon^{*(kl)}_{pq}$ into eqn (7), we obtain the homogenized constitutive parameters. Applying prestrain to the base cell is implemented by subjecting the discretized model to equivalent nodal loads [given by the right-hand side of eqn (8)], and periodic boundary conditions are imposed by a penalty method (Cook *et al.*, 1989).

The described asymptotic homogenization procedure provides rigorous convergence estimates of the displacements in the real structure and the displacements computed with the use of the homogenized coefficients [e.g. Bensoussan *et al.* (1978); Sanchez-Palencia (1980)]. As for the predicted effective material properties, the method corresponds exactly to the energy based approach that employs average stress and strain theorems (Hashin, 1983; Aboudi, 1991). In this method, three tests of the base cell, namely two tensile tests and one shear test, given by $\varepsilon^{0(kl)}_{pq}$ imposed only on the boundaries of the base cell, induce the strain fields $\varepsilon^{A(kl)}_{pq}$. This strain field corresponds to the superimposed strain fields $(\varepsilon^{0(kl)}_{pq} - \varepsilon^{*(kl)}_{pq})$ in eqn (7). In other words, the homogenization method and the energy method give the same results and the only difference lies in the test strain field, which, in the homogenization case, is a superposition of a homogeneous strain field $\varepsilon^{0(kl)}_{pq}$ and a strain field $\varepsilon^{*(kl)}_{pq}$ induced by the inhomogeneities of the base cell and in the energy case is a strain field induced by $\varepsilon^{0(kl)}_{pq}$ imposed only at the boundaries.

As mentioned in the introduction, construction of materials with prescribed constitutive parameters can conveniently be treated as a topology optimization problem. Therefore, expressing the constitutive parameters in terms of element mutual energies makes it possible to use effective existing algorithms used in topology optimization of trusses, plates, grillages, etc. Homogenization expressed in terms of element mutual energies is described in the following; rewriting eqn (7) in terms of mutual energies and using that eqn (8) is satisfied, we get

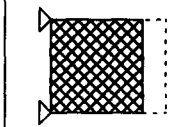
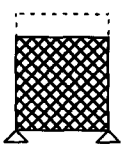
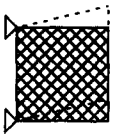
			
$\varepsilon^{0(kl)}_{pq}$	$\varepsilon^{0(11)}_{11} = 1.0$	$\varepsilon^{0(22)}_{11} = 0.0$	$\varepsilon^{0(12)}_{11} = 0.0$
	$\varepsilon^{0(11)}_{22} = 0.0$	$\varepsilon^{0(22)}_{22} = 1.0$	$\varepsilon^{0(12)}_{22} = 0.0$
	$\varepsilon^{0(11)}_{12} = 0.0$	$\varepsilon^{0(22)}_{12} = 0.0$	$\varepsilon^{0(12)}_{12} = 1.0$

Fig. 3. Sketch of the three prestrain cases used in homogenization.

$$E^H_{ijkl} = \frac{1}{Y}\int_Y \left(E_{pqrs}(\varepsilon^{0(kl)}_{pq} - \varepsilon^{*(kl)}_{pq})(\varepsilon^{0(ij)}_{rs} - \varepsilon^{*(ij)}_{rs})\right) \mathrm{d}Y. \tag{9}$$

Assuming that the base cell is discretized by NE finite elements with element areas Y^e, the six independent constitutive parameters can be written in sums of element mutual energies

$$E^H_{1111} = \sum_{e=1}^{NE} Q^e_1, \quad E^H_{2222} = \sum_{e=1}^{NE} Q^e_2, \quad E^H_{1212} = \sum_{e=1}^{NE} Q^e_3,$$

$$E^H_{1122} = \sum_{e=1}^{NE} Q^e_4, \quad E^H_{1112} = \sum_{e=1}^{NE} Q^e_5 \quad \text{and} \quad E^H_{2212} = \sum_{e=1}^{NE} Q^e_6 \tag{10}$$

where the element energies Q^e_I are defined as

$$Q^e_I = \frac{1}{Y^e}\int_{Y^e} \left(E_{pqrs}(\varepsilon^{0(kl)}_{pq} - \varepsilon^{*(kl)}_{pq})(\varepsilon^{0(ij)}_{rs} - \varepsilon^{*(ij)}_{rs})\right) \mathrm{d}Y^e, \quad \begin{cases} e = 1, \ldots, NE \\ I = 1, \ldots, 6 \end{cases} \tag{11}$$

where Y^e is the area of element e and we make use of the abbreviation $I \rightarrow ijkl$,

$$1 \rightarrow 1111, \quad 2 \rightarrow 2222, \quad 3 \rightarrow 1212, \quad 4 \rightarrow 1122, \quad 5 \rightarrow 1112, \quad 6 \rightarrow 2212. \tag{12}$$

4. OPTIMIZATION PROCEDURE

As shown in the last section, the constitutive parameters of a material can be expressed in terms of element mutual energies. This feature makes the optimization problem well suited for optimality criteria (OC) methods. In the following, a modified version of the OC method proposed in Zhou and Rozvany (1993) will be used.

Given the constitutive matrix, we have six equality constraints given as

$$\sum_{e=1}^{NE} Q^e_I - D^*_I = 0, \quad I = 1, 6 \tag{13}$$

where D^*_I are the six prescribed constitutive parameters, and the abbreviation i is given by eqn (12) and the element mutual energies Q^e_I are given by eqn (11).

As cost function we choose the weight of a base cell

$$W = \sum_{e=1}^{NE} \gamma^e x^e, \quad \text{where} \quad \gamma^e = \begin{cases} \varrho_m A^e & \text{(continuum)} \\ \varrho_m l^e \left(\dfrac{l^{\text{pref}}}{l^e}\right)^\eta & \text{(truss).} \end{cases} \tag{14}$$

Here ϱ_m is specific weight, while the other variables are dependent on the type of substructure considered. In the continuum case, the design variable x^e is the thickness (or Young's modulus) of an element and A^e is the area of an element. In the truss case, the design variable x^e is the bar area and l^e is the length of the eth bar. A penalizing term $l^e(l^{\text{pref}}/l^e)^\eta$ is added in the truss case in order to be able to penalize solutions with long or short bars. l^{pref} is a preferred length of the bar members and bars longer or shorter than l^{pref} will be penalized by choosing η positive or negative.

Now the optimization problem can be stated as

$$\text{minimize:} \quad W = \sum_{e=1}^{NE} \gamma^e x^e$$

$$\text{subject to:} \quad \sum_{e=1}^{NE} Q_I^e - D_I^* = 0, \quad I = 1, \ldots, 6$$

$$\text{and:} \quad x_{\min} \leqslant x^e \leqslant x_{\max}, \quad e = 1, \ldots, NE \tag{15}$$

where $x_{\min}$ and $x_{\max}$ are lower and upper limits on the design variables.

The Lagrangian function can be written as

$$\mathscr{L} = \sum_{e=1}^{NE} \gamma^e x^e + \sum_{I=1}^{6} \nu_I \left(D_I^* - \sum_{e=1}^{NE} Q_I^e \right) + \sum_{e=1}^{NE} \alpha^e (-x^e + x_{\min}) + \sum_{e=1}^{NE} \beta^e (x^e - x_{\max}) \tag{16}$$

where ν_I is the six Lagrangian multipliers for the six equality constraints and α^e and β^e are the Lagrangian multipliers for the lower side constraints.

Stationarity with respect to design variable x^e gives

$$\frac{\partial \mathscr{L}}{\partial x^e} = \gamma^e - \sum_{I=1}^{6} \nu_I \frac{Q_I^e}{x^e} - \alpha^e + \beta^e = 0, \quad e = 1, \ldots, NE. \tag{17}$$

Based on eqn (17), the following updating scheme for the design variable x^e can be used:

$$x^e = \begin{cases} x_{\min} & \text{for} \quad \displaystyle\sum_{I=1}^{6} \nu_I \frac{Q_I^e}{\gamma^e} \leqslant x_{\min} \quad \Rightarrow x^e \in R_{\min} \\ \displaystyle\sum_{I=1}^{6} \nu_I \frac{Q_I^e}{\gamma^e} & \text{for} \quad x_{\min} \leqslant \displaystyle\sum_{I=1}^{6} \nu_I \frac{Q_I^e}{\gamma^e} \leqslant x_{\max} \quad \Rightarrow x^e \in R_{\text{act}}, \quad e = 1, \ldots, NE \\ x_{\max} & \text{for} \quad x_{\max} \leqslant \displaystyle\sum_{I=1}^{6} \nu_I \frac{Q_I^e}{\gamma^e} \quad \Rightarrow x^e \in R_{\max} \end{cases} \tag{18}$$

where $R_{\min}$, R_{act} and $R_{\max}$ denote elements governed by the three different updating rules. The six Lagrangian multipliers ν_I are found from the equality constraints by a Newton–Raphson procedure described in the following. The equality constraints (13) can be modified to

$$\sum_{e=1}^{NE} \frac{x^e}{x_0^e} Q_I^e - D_I^* = 0, \quad I = 1, \ldots, 6 \tag{19}$$

where x_0^e is the value of the design variable from the preceding iteration. The factor x^e/x_0^e will be equal to unity when the procedure has converged and will therefore have no effect on the resulting optimal solution. Substituting eqn (18) into eqn (19), we get

$$\phi_I = \sum_{e \in R_{\text{act}}} \frac{Q_I^e \sum_{J=1}^{6} \nu_J Q_J^e}{x_0^e \gamma^e} + \sum_{e \in R_{\min}} \frac{Q_I^e x_{\min}}{x_0^e} + \sum_{e \in R_{\max}} \frac{Q_I^e x_{\max}}{x_0^e} - D_I^* = 0, \quad I = 1, \ldots, 6 \tag{20}$$

where ϕ_I determines the errors in satisfying the Ith equality constraint. The error is minimized by a standard Newton iterative procedure. The recurrence formula is

$$\nu_{k+1} = \nu_k - (\nabla\phi)_k^{-1}\phi_k \tag{21}$$

where k is the iteration number and $\nabla\phi$ is the 6 by 6 Jacobian matrix given as

$$\nabla\phi = \frac{\partial\phi_J}{\partial\nu_I} = \sum_{c\in R_{\mathrm{act}}} \frac{Q_I^c Q_J^c}{X_0^c \nu^c}. \tag{22}$$

Now, one step in the computational procedure can be summarized as

- factorize stiffness matrix
- solve three prestrain cases given by eqn (8) and Fig. 3
- calculate element mutual energies by eqn (11)
- update design variables by eqn (18)
- perform inner loop to update Lagrangian multipliers by eqn (21)
- check convergence.

To stabilize convergence of the optimization algorithm, design changes are constrained by interactively controlled move limits. By limiting changes in design variables to 15% of their initial value, convergence to five digits' accuracy needed approximately 20 iterations for the truss-like materials, but for the continuum-like materials, convergence was generally slow (up to 100 iterations).

As mentioned in Section 2, some of the six constitutive parameters can be dependent or zero. For instance, E_{1112} and E_{2212} are zero for orthotropic materials, which happens when the base cell is symmetric about the horizontal and the vertical axes. In this case, numerical experiments showed that convergence is faster if the two equality constraints for E_{1112} and E_{2212} are replaced by a subroutine which enforces symmetry about the horizontal and vertical axes.

5. EXAMPLES

This section will show a variety of examples demonstrating the capabilities of the proposed method. It will be shown that the "optimal" materials proposed in Bendsøe *et al.* (1993) can be constructed from both truss-like and continuum-like materials, and examples of materials with negative Poisson's ratio will be compared with examples from literature. The examples are presented in a way that enables direct comparison of the truss- and continuum-like materials. The presented results are limited to orthotropic materials with axis of symmetry coaligned with the horizontal and vertical axes, although all kinds of anisotropic and orthotropic materials with principal axes oriented in different directions can be constructed with the proposed algorithm.

Unless otherwise specified, all truss-like cells are constructed from a 4 by 4 node full ground structure (known from topology optimization) with all nodes connected by bars, i.e. there are 120 potential bar members when overlapping is allowed (a 3 by 3 ground structure with 36 potential bars is shown in Fig. 2). Choosing a 4 by 4 node ground structure seems to give the best results. Smaller ground structures do not have enough degrees of freedom to model complicated micromechanical behaviour, and larger ground structures tend to give more complicated topologies as results and are not so easy to interpret physically. It should be noted that all figures showing cells modelled as trusses should show all 120 bar members, but bars governed by the minimum area constraint are not shown in order to make the pictures simpler. Active bars (not governed by the lower side constraint) are shown as lines proportional to their areas and nodes are only shown if they are supporting active bars. The minimum area constraint is set approximately 10^5 times smaller than the maximum areas such that bars governed by this constraint do not have much influence on weight and constitutive parameters, but does imply well-posedness of the homogenization formulation.

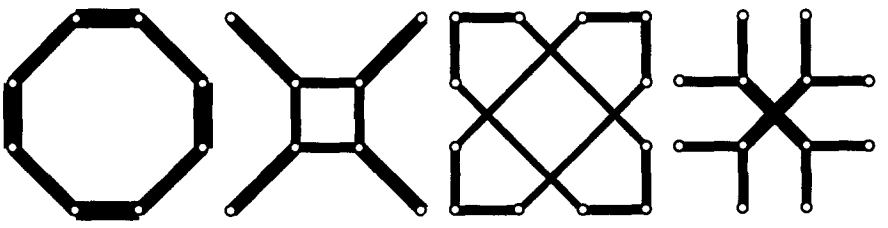
Examples with continuum-like materials are made from square discs with different numbers of four-node bilinear elements. The thickness of each finite element is a design

variable for the optimization algorithm. Using the same discretization for calculation of the homogenized parameters and for the design variable might give rise to inaccurate determination of the homogenized coefficients because of the rapid variation of thicknesses in the base cell. A better calculation of the homogenized coefficients is reached by discretizing the base cell using nine-node biquadratic elements, but as this better approximation did not result in different optimized base cell topologies, discretization by four-node elements was chosen due to computer time requirements. Ideally adaptive remeshing in areas with rapid thickness variation [e.g. Bendsøe and Kikuchi (1988)] should be invoked, but we believe that the resulting base cell topologies will be qualitatively the same as for the simpler discretizations.

The elements are modelled as plane stress elements with Poisson's ratio 0.3 and Young's modulus 0.91 [this value is chosen to make E_{1111} equal to 1 in eqn (3)]. The minimum allowable thickness of an element is chosen to 0.01, and the maximum thicknesses are given in the respective figure texts. In the figures, element thicknesses are represented by linearly varying grey scales with black as maximum thickness and, again, elements governed by the lower side constraints are not shown to make the figures easier to interpret.

The first series of examples will show "optimal" materials (Bendsøe *et al.*, 1993), where the constitutive parameters are given by eqn (4) and ε_I and ε_{II} are given varying values. Figure 4 shows four different topologies of "optimal" materials for $\varepsilon_I = \varepsilon_{II} = 1.0$, which corresponds to a material with Poisson's ratio 1.0 [from eqns (3) and (4)]. All the resulting topologies have the same weight so this is an example of a very flat optimum. An infinite number of other topologies with the same constitutive parameters could be constructed as linear combinations of the four (there could be more!) basic topologies shown. The four different topologies in Fig. 4 were obtained by varying the penalty terms in eqn (14). For instance, the left topology was obtained by penalizing long bars and the second topology from right by penalizing short bars. The two others were obtained by experimenting with different sizes of the move limits, but common to all results was a "switching" between the basic topologies such that move limits had to be made quite small to ensure final convergence. It is seen that the optimal topologies are mechanisms as expected, but it might be difficult to see that they are stable at all. The explanation is that stability comes from the periodicity and is illustrated in Fig. 5. In this figure, periodic materials are constructed from the base cells in Fig. 4 and here it is easier to see that the materials are stable subject to a strain field with equal principal strains oriented vertically/horizontally ($\varepsilon_I = \varepsilon_{II} = 1.0$).

Figure 6 shows two materials with the same constitutive behaviour as the previous example, but this time they are modelled as continuum discs. In the continuum case it was not possible to obtain shear stiffnesses exactly equal to zero because elements governed by the lower side constraint still give noticeable stiffness to the structure. It is interesting, though not surprising, to see the similarity between the continuum topologies in Fig. 6 and the two left-hand (truss) topologies in Fig. 4. This similarity corresponds to the well known similarity between low volume discs and truss topologies in the field of topology optimization [e.g. Rozvany and Zhou (1993)]. In Fig. 7, periodic materials are constructed



$$[E] = \begin{bmatrix} 1.00 & 1.00 & 0.00 \\ 1.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}$$

Fig. 4. "Optimal" materials for $\varepsilon_I = 1.0$ and $\varepsilon_{II} = 1.0$ or Poisson's ratio 1.0 materials. Weight is 4.00 for all.

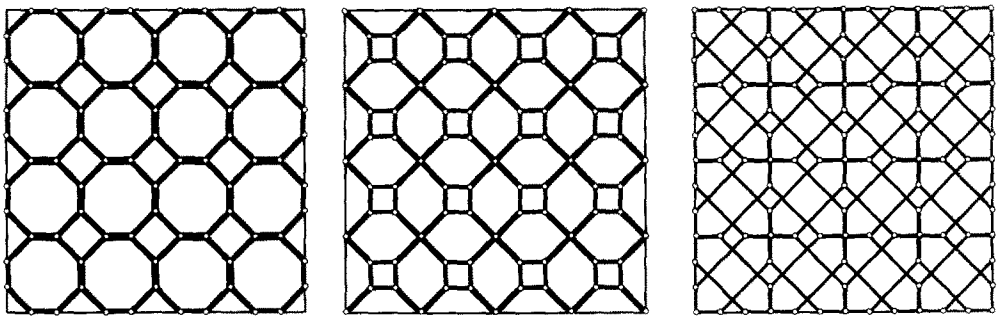
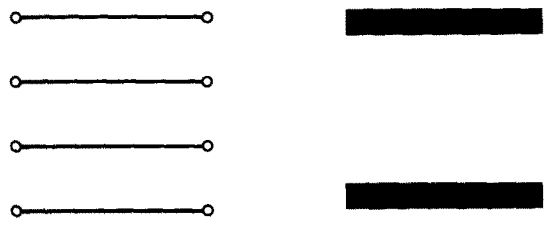


Fig. 5. Periodic materials with Poisson's ratio 1.0 constructed from the first three microstructures in Fig. 4.

from the base cell in Fig. 6, and it is seen that the two materials are very similar except for a diagonal translation of half a period.

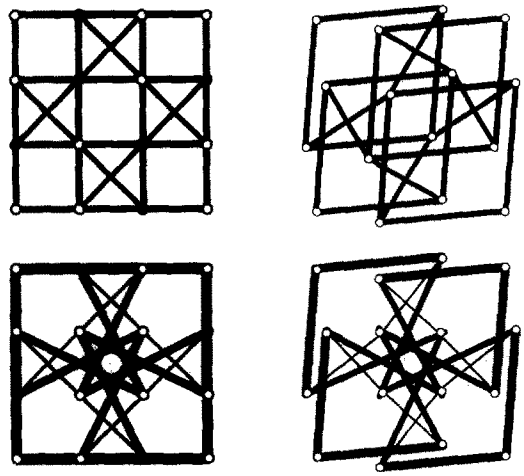
Figure 8 shows "optimal" materials for $\varepsilon_I = 1.0$ and $\varepsilon_{II} = 0.5$ and Fig. 9 shows "optimal" materials for the second principal strain equal to zero. It is obvious that the latter materials only can take up loads in the horizontal direction and that they have zero shear stiffness.

For materials with $E_{1122} < E_{1212}$ the microstructures tend to be somewhat more complex than those previously shown. Figure 10 shows two topologies of truss-like materials for $\varepsilon_I = 1.0$ and $\varepsilon_{II} = -1.0$. The optimal topologies have overlapping bars, so their shear



$$[E] = \begin{bmatrix} 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix} \qquad [E] = \begin{bmatrix} 1.00 & 0.01 & 0.00 \\ 0.01 & 0.01 & 0.00 \\ 0.00 & 0.00 & 0.01 \end{bmatrix}$$

Fig. 9. "Optimal" materials for $\varepsilon_I = 1.0$ and $\varepsilon_{II} = 0.0$. Weights are 1.00 and 1.10, respectively and the continuum-like material at right has maximum thickness 4.45.



$$[E] = \begin{bmatrix} 1.00 & -1.00 & 0.00 \\ -1.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}$$

Fig. 10. "Optimal" material for $\varepsilon_I = 1.0$ and $\varepsilon_{II} = -1.0$. Weights are 7.11 (top left) and 9.04 (bottom left). Figures at right show the respective shear-displacement modes.

$$[E] = \begin{bmatrix} 1.00 & -0.50 & 0.00 \\ -0.50 & 0.25 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}$$

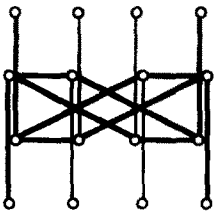
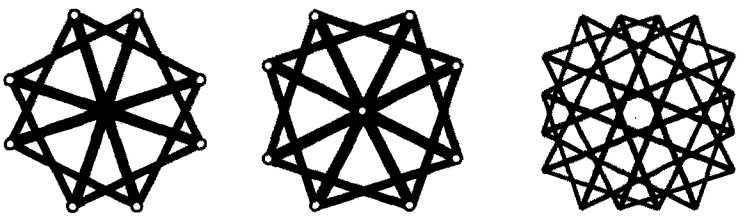


Fig. 12. "Optimal" materials for $\varepsilon_I = 1.0$, $\varepsilon_{II} = -0.5$. Weight is 1.0 (some vertical bars are translated horizontally to illustrate overlapping).

displacement modes ($E_{pq}^{0(12)} = 1.0$ in Fig. 3) are shown to illustrate the overlapping. The materials compress vertically subject to horizontal compression, but in shear they will collapse in the mechanism-type motion shown in the figure. In Fig. 11, a continuum-like material with the same constitutive behaviour is shown. Note that this material is very similar to the zero shear stiffness negative Poisson's ratio material proposed in Almgren (1985). A parallel to Milton (1992) can also be drawn. In this paper, almost rigid inclusions are connected by small square "links". Links of this kind are also observed in Fig. 11. Figure 12 shows an "optimal" truss-like material for $\varepsilon_I = 1.0$ and $\varepsilon_{II} = -0.5$ and, here again, the resulting topology has overlapping bars.

The latest examples showed materials with behaviour similar to Poisson's ratio negative materials except for the shear stiffness (E_{1212}) being zero. The next examples will show "real" negative Poisson's ratio materials, i.e. materials strong in shear. In Fig. 13, three truss-like materials with Poisson's ratio -1.0 are shown. Although constructed from different sizes of ground structures, the three base cells show similar mechanical behaviour. They all consist of "stiff" quadratic frames rotating about the centre, such that expansion in one direction rotates the frames in opposite directions and thereby causes expansion perpendicular to the applied expansion. Subjected to shear, the frames will "lock" because of the periodicity conditions.

Construction of negative Poisson's ratio materials from continuum-like materials has turned out to be very difficult with the proposed method. Extensive numerical experiments including different numbers of elements, sizes of move limits and initial guesses did not produce any satisfying results, but showed that the attainable values of E_{1122} and E_{1212} are connected. This observation can be interpreted as follows. E_{1212} is a measure of the shear stiffness of a material. If we want to make a material with a high positive value of E_{1212}, the material should have a high shear stiffness which at the same time prevents the material from forming mechanism-type topologies that are obviously needed to make materials with $E_{1122} < 0$. Two examples are given in Fig. 14. It was only possible to obtain E_{1212} values of 0.3 and 0.15, respectively, and it is seen that neither of the materials has "linking" elements like those seen in Fig. 11. The materials show some similarities to the foam materials of Lakes (1987), although the micromechanical behaviour of the foam materials



$$[E] = \begin{bmatrix} 1.00 & -1.00 & 0.00 \\ -1.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}$$

Fig. 13. Poisson's ratio -1.0 materials. Weights are 8.89, 10.01 and 9.40 and the ground structure had 120, 252 and 630 members, respectively (links are not shown in the material on the right).

includes nonlinear displacements and contact. The fact that micromechanical behaviour is restricted to linear displacements in this paper could be one reason why negative Poisson's ratio materials are difficult to construct from the continuum-like base cells. A microstructure that almost exhibits the behaviour of a Poisson's ratio -0.2 material is shown in Fig. 15.

The problems in constructing a Poisson's ratio -1.0 material strong in shear from a continuum-like base cell could also be explained by the simple choice of ground structure composed of square elements. For instance, Milton (1992) proposed a Poisson's ratio negative material composed of a hexagonal microstructure, which would be very difficult to model using only regular square elements.

The proposed materials can be criticized for a couple of reasons. The first criticism is that infinite stresses will occur in some of the proposed continuum-like materials, because the rigid parts are only being connected corner to corner. This problem can be solved by forming a "smooth" bridge between the rigid faces as proposed by Milton (1992). Another approach, which should be done anyway, is to take the proposed topologies as inputs to a shape optimization program which would smooth the sharp corners, make manufacturing easier and eliminate stress concentrations. The resulting smooth topologies can be manufactured by laser cutting or punching discs of stiff material. To make the discs nonporous, the laser cut microstructures can be embedded in a weak matrix material.

A second criticism is that the proposed truss-like materials will be very costly to manufacture with their complicated systems of overlapping bars. In fact, manufacturing might not be so complicated; take, for example, the Poisson's ratio -1.0 microstructures from Fig. 13. The two microstructures on the left consist of two overlapping frames rotating about their midpoints. The material could be constructed as a laminate consisting of two punched or laser cut discs shown in dark grey and black, respectively, in Fig. 16. The two layers could be embedded in a soft matrix material and possibly be connected by small pins as sketched in the figure. Embedding the microstructure in a soft matrix material would naturally cause the Poisson's ratio to become higher than -1.0, depending on the stiffness ratio between the strong microstructure and the weak matrix type material.

Another interesting possibility to construct the proposed materials lies in the field of polymer and crystal growth. It might (in future) be possible to "grow" polymers and crystals on a microscopic scale and thereby produce "artificial" materials which exhibit given constitutive behaviour.

6. CONCLUSIONS

An efficient procedure has ben proposed, for constructing materials with any prescribed constitutive tensor only restricted to be positive semi-definite. The constitutive parameters of periodic materials with microstructures modelled as two-dimensional trusses or continuums are found by a numerical homogenization method expressed in terms of element mutual energies. The construction problem is formulated as an inverse problem of finding the lightest microstructure for the given constitutive parameters. The proposed method uses a ground structure approach known from topology optimization, where areas of bars in a truss-like base cell or element thicknesss in a continuum-like base cell are used as design variables. The inverse problem is solved by a multiple constraint optimality criteria method, where weight of the base cell is the cost function to be minimized and constraints are the six given constitutive parameters.

Numerical results show that materials with arbitrary constitutive parameters, including the "optimal" materials proposed in Bendsøe *et al.* (1993) and Poisson's ratio -1.0 materials, can be constructed from truss-like base cells. Furthermore, a wide range of materials can be constructed from continuum-like base cells, though it seems to be difficult to obtain negative Poisson's ratio materials from the simple ground structures of square elements used in this paper. Most of the proposed materials can be readily manufactured by modern punching or laser cutting methods and a suggestion is given, of how to manufacture a material with Poisson's ratio -1.0 using two layers embedded in a weak matrix material.

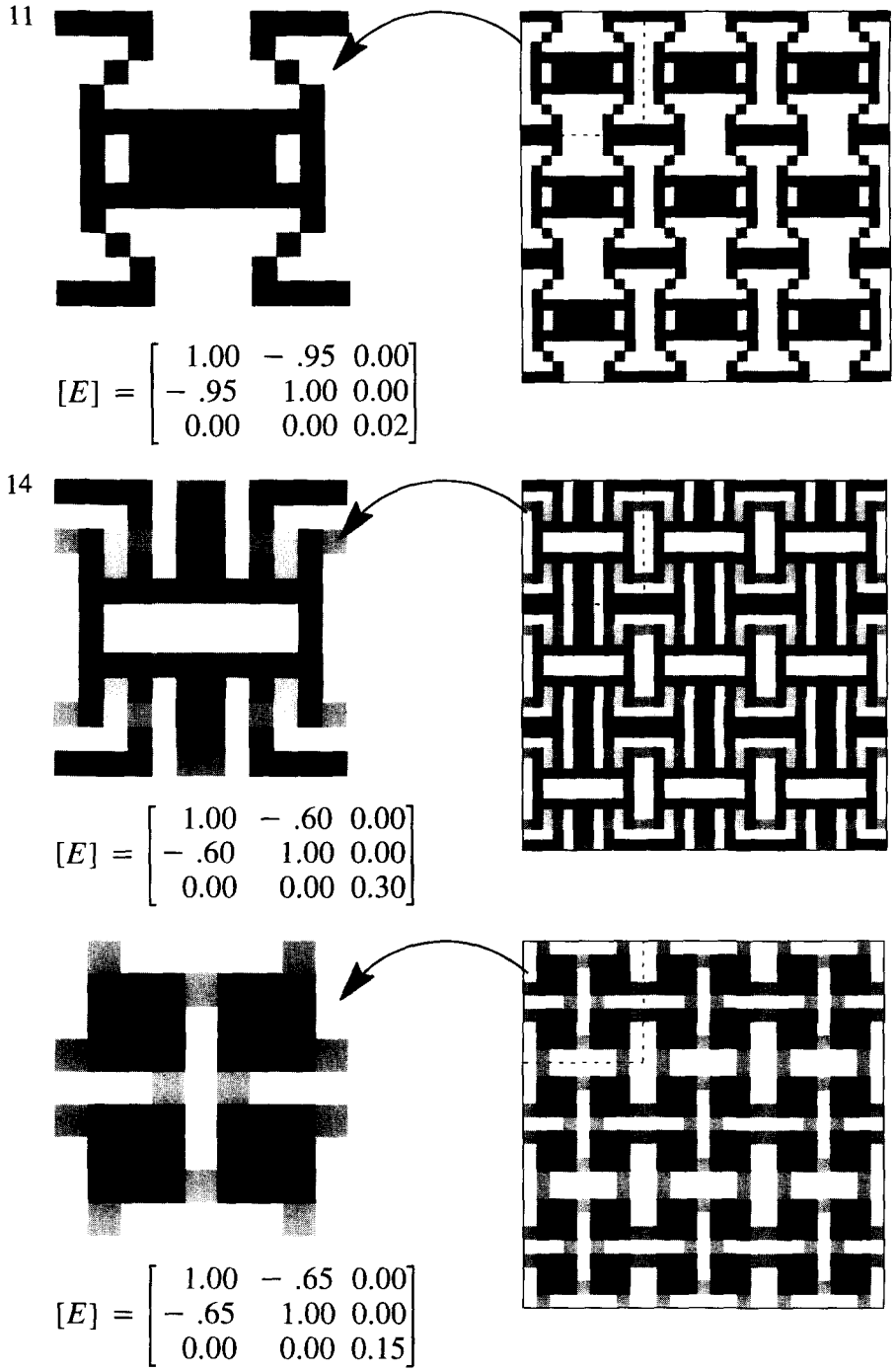
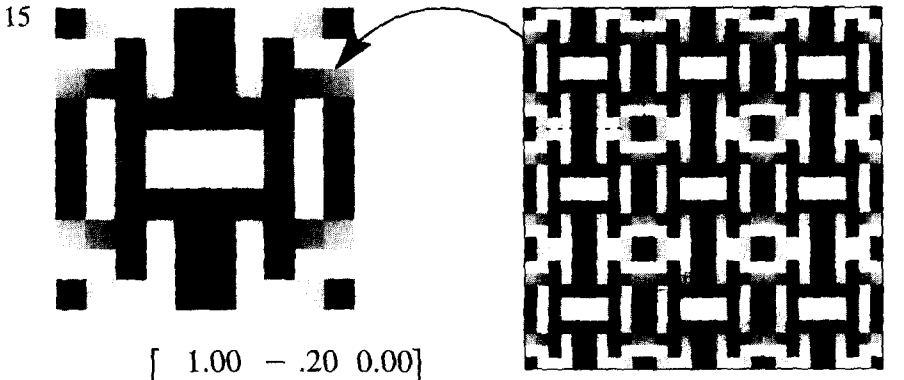


Fig. 11. "Optimal" material for $\varepsilon_I = 1.0$ and $\varepsilon_{II} = -1.0$. Weight is 13.87 and maximum thickness is 40.0.

Fig. 14. Materials exhibiting negative Poisson's ratio behaviour though weak in shear. Weights are 9.75 and 4.14 and maximum thicknesses are 30.0 and 10.0, respectively.



$$[E] = \begin{bmatrix} 1.00 & -.20 & 0.00 \\ -.20 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.40 \end{bmatrix}$$

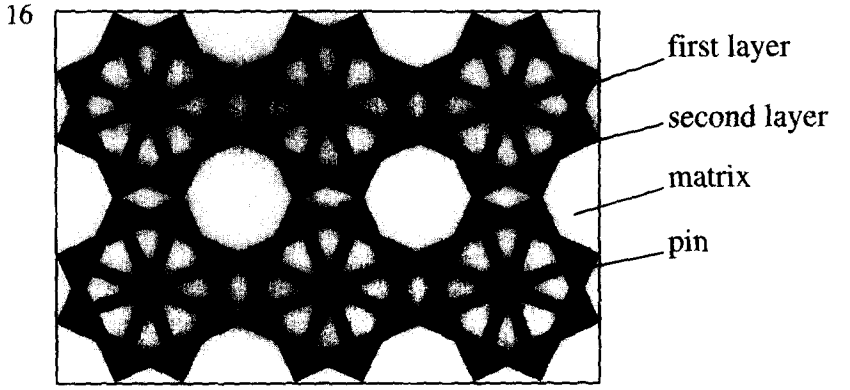


Fig. 15. Material with Poisson's ratio near -0.2 though too weak in shear. Weight is 8.22 and maximum thicknes is 20.0.

Fig. 16. Sketch of practical implementation of the proposed materials. Poisson's ratio -1.0 material from microstructures in Fig. 13 constructed as a material with two layers interconnected by small pins (top layer dark grey, bottom layer black and weak matrix material grey).

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